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1976 J. Phys. A: Math. Gen. 9 L1

(http://iopscience.iop.org/0305-4470/9/1/001)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

LETTER TO THE EDITOR

Proof of interesting relations involving 3-*j* symbols

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Received 12 November 1975

Abstract. Direct proofs of a 'pseudo-orthogonality' relation amongst the 3-*j* symbols and vanishing of a more complicated summation conjectured before are provided.

Morgan (1975) has obtained a pseudo-orthogonality relation

$$S_l=\sum_{l'=0}^{l}\frac{1}{2l'-1}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0\end{pmatrix}^2=\begin{cases}-1 & \text{for } l=0\\ 0 & \text{for } l\geqslant 1\end{cases} \tag{1}$$

and has conjectured that

$$\bar{S}_l=\sum_{l'=0}^{l}\frac{1}{2l'+3}\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0\end{pmatrix}^2-\sum_{l'=0}^{l}\frac{1}{2l'+1}\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0\end{pmatrix}^2=0. \tag{2}$$

He has given a recursive proof of (1) but could not prove (2) for any natural number l.

In the following, we shall prove both (1) and (2) directly using the values of the 3-j symbols involved in these relations. From the definition of the 3-j symbols in terms of the Clebsch–Gordan coefficients for which a general formula is known (equations (3.7.3) and (3.6.11) in Edmonds 1960), we can immediately derive

$$\begin{pmatrix} l & l' & l-l' \\ 0 & 0 & 0\end{pmatrix}^2=\frac{(l!)^2(2l')!(2l-2l')!}{(2l+1)!(l'!)^2[(l-l')!]^2} \tag{3}$$

$$\begin{pmatrix} l & l'+1 & l-l'+1 \\ 0 & 0 & 0\end{pmatrix}^2=\frac{2[(l+1)!]^2(2l')!(2l-2l')!}{(2l+3)!(l'!)^2[(l-l')!]^2}. \tag{4}$$

Now we use the duplication formula

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+1) \tag{5}$$

for the gamma functions $(2l')!=\Gamma(2l'+1)$ and $(2l-2l')!=\Gamma(2l-2l'+1)$ in (3) and (4) and substitute the simplified results in (1) and (2) to arrive at

$$S_l=\frac{(l!)^2 2^{2l-1}}{\pi(2l+1)!}\sum_{l'=0}^{l}\frac{\Gamma(l'-\frac{1}{2})\Gamma(l-l'+\frac{1}{2})}{l'!\Gamma(l-l'+1)} \tag{6}$$

$$\bar{S}_l=\frac{(l!)^2(2l+2)2^{2l-1}}{\pi(2l+3)!}\sum_{l'=0}^{l}\frac{\Gamma(l'+\frac{1}{2})\Gamma(l-l'+\frac{1}{2})}{l'!\Gamma(l-l'+1)}\left((2l+3)\frac{\Gamma(l'+\frac{3}{2})}{\Gamma(l'+\frac{5}{2})}-(l+1)\frac{\Gamma(l'+\frac{1}{2})}{\Gamma(l'+\frac{3}{2})}\right) \tag{7}$$

where we have used $\Gamma(z+1)=z\Gamma(z)$. This formula is repeatedly used afterwards also.

Next we attempt to express the sums in (6) and (7) in the form of the hypergeometric functions of argument 1. For this purpose, we first write

$$\frac{\Gamma(l-l'+\frac{1}{2})}{\Gamma(l-l'+1)}=\frac{\sin\pi(l-l'+1)}{\sin\pi(l-l'+\frac{1}{2})}\frac{\Gamma(l'-l)}{\Gamma(l'+\frac{1}{2}-l)}=\frac{\sin\pi l}{\sin\pi(l-\frac{1}{2})}\frac{\Gamma(l'-l)}{\Gamma(l'+\frac{1}{2}-l)} \tag{8}$$

which gives

$$\begin{aligned}\sum_{l'=0}^{l}\frac{\Gamma(l'-\frac{1}{2})\Gamma(l-l'+\frac{1}{2})}{l'!\Gamma(l-l'+1)} &=\frac{\sin\pi l}{\sin\pi(l-\frac{1}{2})}\sum_{l'=0}^{l}\frac{\Gamma(l'-\frac{1}{2})\Gamma(l'-l)}{l'!\Gamma(l'+\frac{1}{2}-l)}\\ &=\frac{\sin\pi l}{\sin\pi(l-\frac{1}{2})}\frac{\Gamma(-\frac{1}{2})\Gamma(-l)}{\Gamma(\frac{1}{2}-l)}\,{}_2F_1(-\tfrac{1}{2},-l;\tfrac{1}{2}-l;1)\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\sum_{l'=0}^{l}\frac{\Gamma(l'+\frac{1}{2})\Gamma(l-l'+\frac{1}{2})}{l'!\Gamma(l-l'+1)}\left((2l+3)\frac{\Gamma(l'+\frac{3}{2})}{\Gamma(l'+\frac{5}{2})}-(l+1)\frac{\Gamma(l'+\frac{1}{2})}{\Gamma(l'+\frac{3}{2})}\right)\\ &\quad=\frac{\sin\pi l}{\sin\pi(l-\frac{1}{2})}\frac{\Gamma(\frac{1}{2})\Gamma(-l)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-l)\Gamma(\frac{3}{2})}\\ &\quad\times\left(\frac{2l+3}{3}\,{}_3F_2(\tfrac{1}{2},\tfrac{3}{2},-l;\tfrac{5}{2},\tfrac{1}{2}-l;1)-(l+1)\,{}_3F_2(\tfrac{1}{2},\tfrac{1}{2},-l;\tfrac{3}{2},\tfrac{1}{2}-l;1)\right).\end{aligned} \tag{10}$$

The ${}_3F_2$ functions in (10) are Saalschützian and can be expressed as products of gamma functions. The ${}_2F_1$ in (9) can also be expressed in the same form (equations 4.4(3) and 2.8(46) in Bateman 1953). Thus we have

$$ {}_2F_1(-\tfrac{1}{2},-l;\tfrac{1}{2}-l;1)=\frac{\Gamma(\frac{1}{2}-l)\Gamma(1)}{\Gamma(1-l)\Gamma(\frac{1}{2})} \tag{11}$$

$$ {}_3F_2(\tfrac{1}{2},\tfrac{3}{2},-l;\tfrac{5}{2},\tfrac{1}{2}-l;1)=\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})\Gamma(2+l)\Gamma(1+l)}{\Gamma(2)\Gamma(1)\Gamma(\frac{5}{2}+l)\Gamma(\frac{1}{2}+l)} \tag{12}$$

$$ {}_3F_2(\tfrac{1}{2},\tfrac{1}{2},-l;\tfrac{3}{2},\tfrac{1}{2}-l;1)=\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})\Gamma(1+l)\Gamma(1+l)}{\Gamma(1)\Gamma(1)\Gamma(\frac{3}{2}+l)\Gamma(\frac{1}{2}+l)} \tag{13}$$

or

$$ {}_3F_2(\tfrac{1}{2},\tfrac{3}{2},-l;\tfrac{5}{2},\tfrac{1}{2}-l;1)=\frac{\frac{3}{2}}{l+\frac{3}{2}}(l+1)\,{}_3F_2(\tfrac{1}{2},\tfrac{1}{2},-l;\tfrac{3}{2},\tfrac{1}{2}-l;1). \tag{14}$$

From (6), (9) and (11), we arrive at

$$S_l=\frac{(l!)^2 2^{2l-1}\sin\pi l\,\Gamma(-\frac{1}{2})}{\pi(2l+1)!\sin\pi(l-\frac{1}{2})(-l)\Gamma(\frac{1}{2})}$$

which is zero when $l\neq 0$, whereas for $l=0$, it is -1 since

$$\lim_{l\to 0}\left(-\frac{\sin\pi l}{l\sin\pi(l-\frac{1}{2})}\right)=\pi \qquad\text{and}\qquad \frac{\Gamma(-\frac{1}{2})}{2\Gamma(\frac{1}{2})}=-1.$$

Again from (7), (10) and (14), we obtain

$$\bar{S}_l = \frac{(l!)^2(2l+2)2^{2l-1}\sin \pi l\,\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(-l)}{\pi(2l+3)!\sin \pi(l-\frac{1}{2})\Gamma(\frac{1}{2}-l)\Gamma(\frac{3}{2})}\times 0.$$

To show now that $\bar{S}_l$ vanishes for every integral $l \geq 0$, we have simply to show that the factor multiplying zero above is finite for every such value of l. Indeed

$$\sin \pi l\,\Gamma(-l) = \frac{\pi \sin \pi l}{\sin(-\pi l)\Gamma(1+l)} = -\frac{\pi}{\Gamma(1+l)} = -\frac{\pi}{l!}$$

which is finite. This establishes that $\bar{S}_l = 0$ for all $l \geq 0$.

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